

The structure of graphs with forbidden C_4 , \overline{C}_4 , C_5 , chair and co-chair

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Abstract

We find the structure of graphs that have no C_4 , \overline{C}_4 , C_5 , chair and co-chair as induced subgraphs.

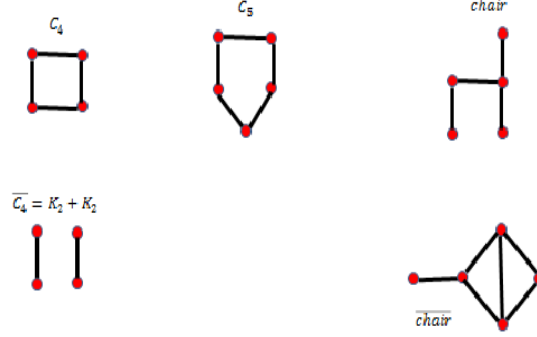
1 Introduction

In this paper, graphs are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Two edges of a graph G are said to be adjacent if they have a common endpoint and two vertices x and y are said to be adjacent if xy is an edge of G . The *neighborhood* of a vertex v in a graph G , denoted by $N_G(v)$, is the set of all vertices adjacent to v and its *degree* is $d_G(v) = |N_G(v)|$. We omit the subscript if the graph is clear from the context. For two set of vertices U and W of a graph G , let $E[U, W]$ denote the set of all edges in the graph G that joins a vertex in U to a vertex in W . A graph is empty if it has no edges. For $A \subseteq V(G)$, $G[A]$ denotes the sub-graph of G induced by A . If $G[A]$ is an empty graph, then A is called a stable. While, if $G[A]$ is a complete graph, then A is called a clique set, that is any two distinct vertices in A are adjacent. The complement graph of G is denoted by \overline{G} and defined as follows: $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$.

A graph H is called forbidden subgraph of G if H is not (isomorphic to) an induced subgraph of G .

A cycle on n vertices is denoted by $C_n = v_1v_2...v_nv_1$ while a path on n vertices is denoted by $P_n = v_1v_2...v_n$. A chair is any graph on 5 distinct vertices x, y, z, t, v with exactly 5 edges xy, yz, zt and zv . The co-chair or *chair* is the complement of a chair (see the below figure).

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Many graphs encountered in the study of graph theory are characterized by configurations or subgraphs they contain. However, there are occasions where it is easier to characterize graphs by sub-graphs or induced sub-graphs they do not contain. For example, trees are the connected graph without (induced) cycles. Bipartite graphs are those without (induced) odd cycles ([5]). Split graphs are those without induced C_4 , $\overline{C_4}$ and C_5 . Line graphs are characterized by the absence of only nine particular graphs as induced sub-graph (see [4]). Perfect graphs are characterized by C_{2n+1} and $\overline{C_{2n+1}}$ being forbidden, for all $n \geq 2$ (see [3]). The purpose of this paper is to find the structure of graphs such that C_4 , $\overline{C_4}$, C_5 , chair and co-chair are forbidden subgraphs.

2 Preliminary Definitions and Theorems

Definition 1. A graph G is called a split graph if its vertex set is the disjoint union of a stable set S and a clique set K . In this case, G is called an $\{S, K\}$ -split graph.

If G is an $\{S, K\}$ -split graph and $\forall s \in S, \forall x \in K$ we have $sx \in E(G)$, then G is called a complete split graph.

If G is an $\{S, K\}$ -split graph and $E[S, K]$ forms a perfect matching of G , then G is called a perfect split graph.

Theorem 1. (Földes and Hammer [1]) G is a split graph if and only if C_4 , $\overline{C_4}$ and C_5 are forbidden subgraphs of G .

Definition 2. ([2]) A threshold graph G can be defined as follows:

- 1) $V(G) = \bigcup_{i=1}^{n+1} (X_i \cup A_{i-1})$, where the A_i 's and X_i 's are pair-wisely disjoint sets.
- 2) $K := \bigcup_{i=1}^{n+1} X_i$ is a clique and the X_i 's are nonempty, except possibly X_{n+1} .

- 3) $S := \bigcup_{i=0}^n A_i$ is a stable set and the A_i 's are nonempty, except possibly A_0 .
- 4) $\forall 1 \leq j \leq i \leq n$, $G[A_i \cup X_j]$ is a complete split graph.
- 5) The only edges of G are the edges of the subgraphs mentioned above.

In this case, G is called an $\{S, K\}$ -threshold graph.

Theorem 2. (Hammer and Chvátal [2]) G is a threshold graph if and only if C_4 , \overline{C}_4 and P_4 are forbidden subgraphs of G .

3 Main Results

Lemma 1. Suppose that C_4 , \overline{C}_4 , C_5 , chair and co-chair are forbidden subgraphs of G . If the path $mbb'm'$ is an induced subgraph of G , then:

$$N(m) - \{b\} = N(m') - \{b'\}$$

and

$$N(b) - \{m\} = N(b') - \{m'\}.$$

Proof. Since C_4 , \overline{C}_4 and C_5 are forbidden, then G is an $\{S, K\}$ -split graph for some stable set S and a clique set K . Since $mbb'm'$ is an induced subgraph of G , then $m, m' \in S$ and $b, b' \in K$.

Assume that there is $x \in N(m) - \{b\}$ but $x \notin N(m') - \{b'\}$. Since xm is an edge of G and S is stable, then we must have $x \in K$. But K is a clique, then x is adjacent to b and b' . Thus $G[\{x, m, b, b', m'\}]$ is a co-chair. Contradiction. So $N(m) - \{b\} \subseteq N(m') - \{b'\}$. By symmetry, $N(m') - \{b'\} \subseteq N(m) - \{b\}$. Thus $N(m) - \{b\} = N(m') - \{b'\}$.

Assume that there is $x \in N(b) - \{m\}$ but $x \notin N(b') - \{m'\}$. Suppose that $x \in S$. Then $G[\{x, m, b, b', m'\}]$ is a chair. Contradiction. Thus $x \in K$. But K is a clique. Whence $x \in N(b') - \{m'\}$. Thus $N(b) - \{m\} \subseteq N(b') - \{m'\}$. By symmetry, $N(b') - \{m'\} \subseteq N(b) - \{m\}$. Therefore $N(b) - \{m\} = N(b') - \{m'\}$. \square

Proposition 1. If P_4 is a forbidden subgraph of an $\{S, K\}$ -split graph G , then G is an $\{S, K\}$ -threshold graph.

Proof. We prove this by induction on the number of vertices of G . This is clearly true for small graphs. Suppose that P_4 is a forbidden subgraph of an $\{S, K\}$ -split graph G . It is clear that G is a threshold graph. We have to prove that G is $\{S, K\}$ -threshold graph. Let $x \in K$ be a vertex with minimum degree in G , that is $d_G(x) = \min\{d_G(y); y \in K\}$ and $G' := G - x$ be the graph induced by the vertices of G except x (If $K = \emptyset$, then the statement is true). Then P_4 is a forbidden subgraph of the $\{S, K - \{x\}\}$ -split graph G' . By the induction hypothesis, G' is an $\{S, K - \{x\}\}$ -threshold graph. We follow the notations in Definition 2. Assume that $\exists a \in S - A_n$ such that $ax \in E(G)$. Let $x_n \in X_n$. Since $d(x_n) \geq d(x)$, then there is $a_n \in A_n$ such that $a_n x_n \in E(G)$ but $a_n x \notin E(G)$. Then $axx_n a_n$ is an induced P_4 in G . Contradiction. Thus we

may suppose that $N(x) \cap S \subseteq A_n$. If $N(x) \cap A_n = \phi$, then we add x to X_{n+1} . If $N(x) \cap A_n = A_n$, then we add x_n to X_n . Otherwise $\phi \subsetneq N(x) \cap A_n \subsetneq A_n$. In this case we do the following: remove from A_n the element of $N(x) \cap A_n$, create $A_{n+1} = N(x) \cap A_n$, remove the elements of X_{n+1} to the new set X_{n+2} and add x to X_{n+1} (so that the new $X_{n+1} = \{x\}$). Then G is $\{S, K\}$ -threshold graph \square

Definition 3. A graph G is called a *comb* if:

- 1) $V(G)$ is disjoint union of sets $A_0, \dots, A_n, M_1, \dots, M_l, X_1, \dots, X_{n+1}, Y_2, \dots, Y_{l+2}$.
Let $Y_1 = X_1$ (These sets are called the sets of the comb G).
- 2) $S := A \cup M$ is a stable set, where $M = \bigcup_{i=1}^l M_i$ and $A = \bigcup_{i=0}^n A_i$
- 3) $K := X \cup Y$ is a clique, where $X = \bigcup_{i=1}^{n+1} X_i$ and $Y = \bigcup_{i=1}^{l+2} Y_i$.
- 4) $\forall 1 \leq j \leq i \leq n, G[A_i \cup X_j]$ is a complete split graph.
- 5) $G[A \cup Y]$ is a complete split graph.
- 6) $\forall 1 \leq i \leq l, G[Y_i \cup M_i]$ is a perfect split graph.
- 7) $\forall 1 \leq i < j \leq l, G[Y_j \cup M_i]$ is a complete split graph.
- 8) $\exists 1 \leq k_0 \leq l, \forall i \leq k_0, G[Y_{l+1} \cup M_i]$ is a complete split graph.
- 9) $X_{n+1}, Y_{l+2}, Y_{l+1}, M_l$ and A_0 are the only possibly empty sets.
- 10) The only edges of G are the edges of the subgraphs mentioned above.

In this case, we say that G is an $\{S, K\}$ -comb.

Lemma 2. Every $\{S, K\}$ -threshold graph is an $\{S, K\}$ -comb.

Proof. Let G be an $\{S, K\}$ -threshold graph defined as in Definition 2. Following the notations in Definition 3, we take $l = 1$ and $M_l = Y_{l+1} = Y_{l+2} = \phi$. This shows that G is an $\{S, K\}$ -comb. \square

Theorem 3. If chair and co-chair are forbidden subgraphs of an $\{S, K\}$ -split graph G , then G is an $\{S, K\}$ -comb.

Proof. We prove the statement by induction on the number of vertices. The statement is true for small graphs. Suppose that chair and co-chair are forbidden subgraphs of an $\{S, K\}$ -split graph G . If P_4 is also a forbidden subgraph of G , then G is an $\{S, K\}$ -threshold graph, and hence, G is an $\{S, K\}$ -comb. So we may suppose that G contains an induced path $abb'a'$. Then $N(a) - \{b\} = N(a') - \{b'\}$ and $N(b) - \{a\} = N(b') - \{a'\}$. Let $S' = S - a'$, $K' = K - b'$ and $G' = G[S' \cup K']$. Then chair and co-chair are forbidden subgraphs of the $\{S', K'\}$ -split graph G' . Then G' is an $\{S', K'\}$ -comb with $S' = A \cup M$ and $K' = X \cup Y$ (we follow the notations as in Definition 3).

If $a \in S'$ and $b \in K'$, then we add a' to the set of the comb G' that contains a and b' to the set of the comb G' that contains b . Thus G is $\{S, K\}$ -comb.

Otherwise, $a \in K$ while $b \in S$. First we suppose that $n \geq 1$. Then there is $x \in A_1$ because $A_1 \neq \phi$. We have the following cases:

case 1: assume that $a \in Y$ and $b \in M$. Then $xabb'a'x$ is an induced C_5 in G . Contradiction.

case 2: assume that $a \in X_i$ and $b \in A_j$. Then by definition of comb, we have $i \leq j$. Then $xabb'a'x$ is an induced C_5 in G . Contradiction. So $i = j$. Assume that there is $y \in \bigcup_{t=i}^n A_t - \{b\}$. Then $yaba'b'y$ is an induced C_5 in G .

Contradiction. Thus we must have $i = n$ and $A_i = A_n = \{b\}$. Assume that there is $y \in X_{n+1}$. Then $yaba'b'y$ is an induced C_5 in G . Contradiction. Thus we must have $X_{n+1} = \phi$. In this case, we do the following: remove a from X_n and add it to A_n , remove b from A_n and add it to X_n , add b' to X_{n+1} , create $A_{n+1} = \{a'\}$ and $X_{n+2} = \phi$. Thus G is an $\{S, K\}$ -comb.

case 3: assume that $a \in X_i$ and $b \in M_j$. Then by the definition of a comb, we must have $i = 1 = j$. But this is already discussed in case 1, because $X_1 = Y_1$.

case 4: Assume that $a \in Y_i$ and $b \in A_j$. The case when $i = 1$ is already discussed in case 2. So we may assume that $i > 1$. Let $y \in M_1$. Then $yaba'b'y$ is an induced C_5 in G . Contradiction.

Second, suppose that $n = 0$. That is $A = A_0$ and so there is no A_1 and no X_2 . We have the following cases:

case 1: Assume that $a \in Y_i$ and $b \in M_i$. If $i > 1$ or $Y_i \neq \{b\}$, then there is $c \in \bigcup_{t=1}^i A_t - \{a\}$. Then $cabb'a'c$ is an induced C_5 in G . Contradiction. Thus $i = 1$ and $Y_1 = \{a\}$. Hence $M_1 = \{b\}$. We can do the following: remove a from Y_1 and add it to M_1 , remove b from M_1 and add it Y_1 , add b' to Y_1 and add a' to M_1 . Thus G is an $\{S, K\}$ -comb.

case 2: Assume that $a \in Y_i$ and $b \in M_j$ with $i > j$. There exist $c \in Y_j$ such that cb is an edge of G . If there is $y \in N_{G'}(a) - N_{G'}(b)$, then $yabb'a'y$ is an induced C_5 in G . Contradiction. Thus, we must have $j = 1$, $Y_1 = \{c\}$, $M_1 = \{b\}$, $i = 2$ and $M_2 = \phi$. We can do the following: remove a from Y_2 and add it to M_1 , remove b from M_1 and add it Y_1 and remove c from Y_1 and add it to Y_2 . Thus G is an $\{S, K\}$ -comb.

case 3: $a \in Y_i$ and $b \in M_j$ with $i < j$. This case is impossible by the definition of the comb.

□

Corollary 1. G is a comb if and only if C_4 , \overline{C}_4 , C_5 , chair and co-chair are forbidden subgraphs of G .

Proof. The necessary condition is obvious by the definition of a comb. For the sufficient condition it is enough to note that the statement C_4 , \overline{C}_4 , C_5 , chair and co-chair are forbidden subgraphs of G is equivalent to the statement that G is a split graph and chair and co-chair are forbidden subgraphs of G . \square

Corollary 2. *G is a comb if and only if \overline{G} is a comb.*

Proof. Enough to note that the complement of C_4 , \overline{C}_4 , C_5 , chair and co-chair are \overline{C}_4 , C_4 , C_5 , co-chair and chair. \square

Corollary 3. *G is a comb if and only if every induced subgraph of G is a comb.*

References

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